Parameter estimation from the one-body density

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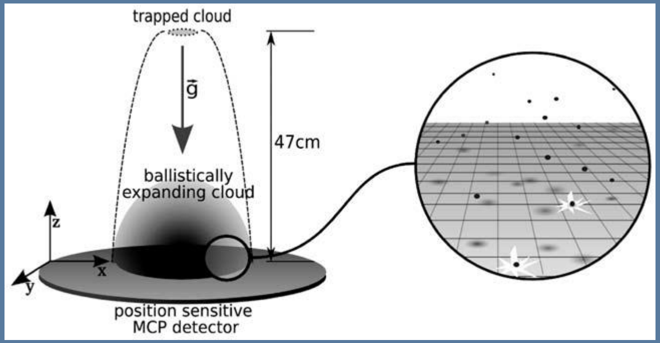
Outline

1. Atom position measurements - motivation
2. Parameter estimation from the density
3. Known example: Mach-Zehnder Interferometer
4. Estimation using the interference pattern
5. Estimation of the temperature of quasi BEC

Atom position measurements

Microchannel plate



Palaiseau

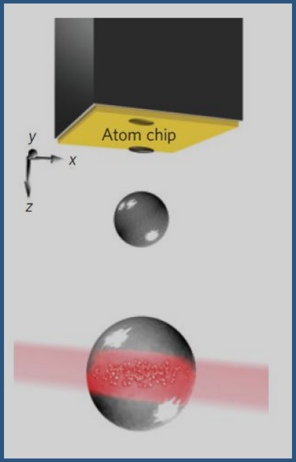
Schellenkens, Science 2005

Perrin, PRL 2007

Jaskula, PRL 2010

many more…

Light-sheet



Vienna

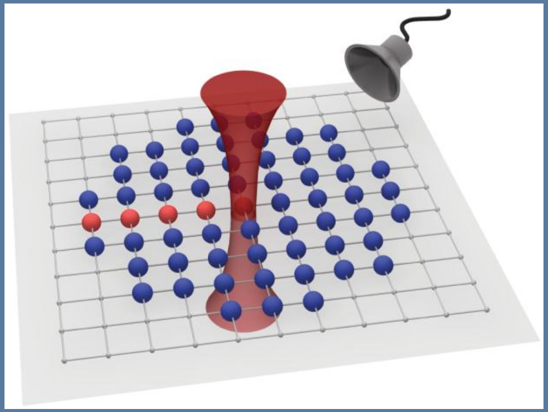
Brucker, NJP 2009

Betz, PRL 2011

Perrin, Nat. Phys. 2012

many more…

Lattice / Mott



Garching

Weitenberg, Nature 2011

Weitenberg, PRL 2011

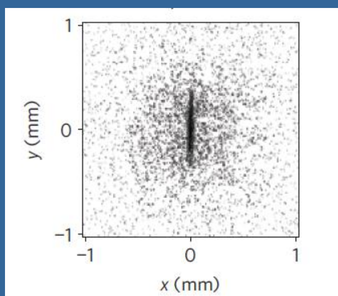
Endres, Science 2011

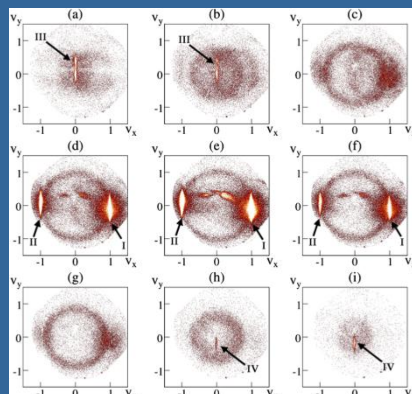
many more…

Some results

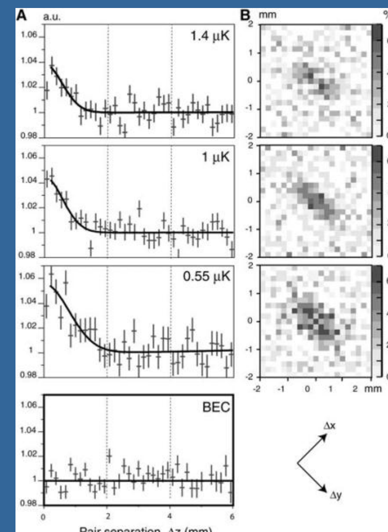
With these modern techniques…

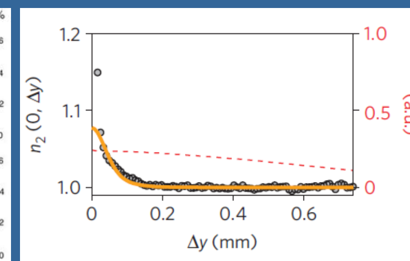
- measure density



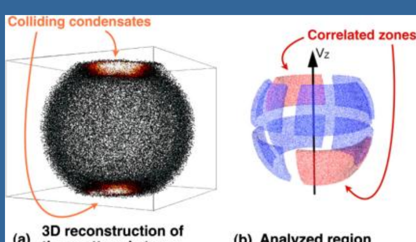


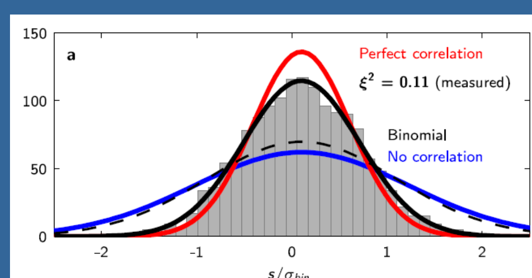
- measure G(2)



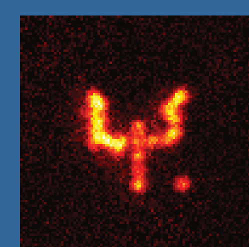


- measure number-squeezing



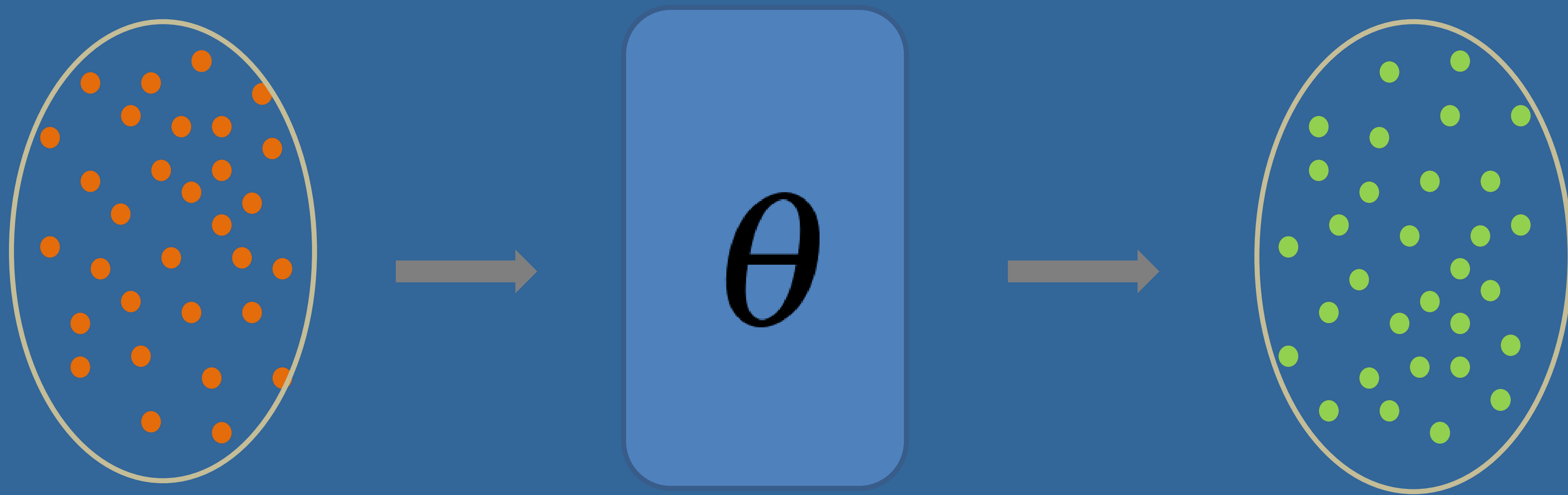


do art…



Parameter estimation

Input state — Unitary evolution — Output state



$|\psi_{\text{in}}\rangle$ — $\hat{U}(\theta)|\psi_{\text{in}}\rangle$ — $|\psi_{\text{out}}(\theta)\rangle$

Measure something and guess $\boldsymbol{\theta}$

Key quantity: <u>precision</u> $\Delta^2\theta$

Estimation from the density

Assumption: the density is known

$$\hat{\Psi}^\dagger(x)\hat{\Psi}(x)$$

In the experiment:

Properties and precision

Crucial: consistency of the estimator $\theta_{\mathrm{ML}} \xrightarrow{m\to\infty} \theta$

Don't need to know separate positions:

Least square fit is equivalent!

PRECISION

$$\mathscr{L}(\varphi) = \prod_{i=1}^{m}\prod_{k=1}^{N} \rho(x_k^{(i)}|\varphi)$$

repeat many times

$$\mathscr{L}(\varphi) = \prod_{i=1}^{m}\prod_{k=1}^{N} \rho(x_k^{(i)}|\varphi)$$

$\downarrow$ $\theta_{\mathrm{ML}}^{(1)}$ $\longrightarrow$ Histogram $\longleftarrow$ $\theta_{\mathrm{ML}}^{(\mathrm{many})}$

$\downarrow$

Width = sensitivity $\Delta\theta_{\mathrm{ML}}$

Calculation of the sensitivity

When m is large:

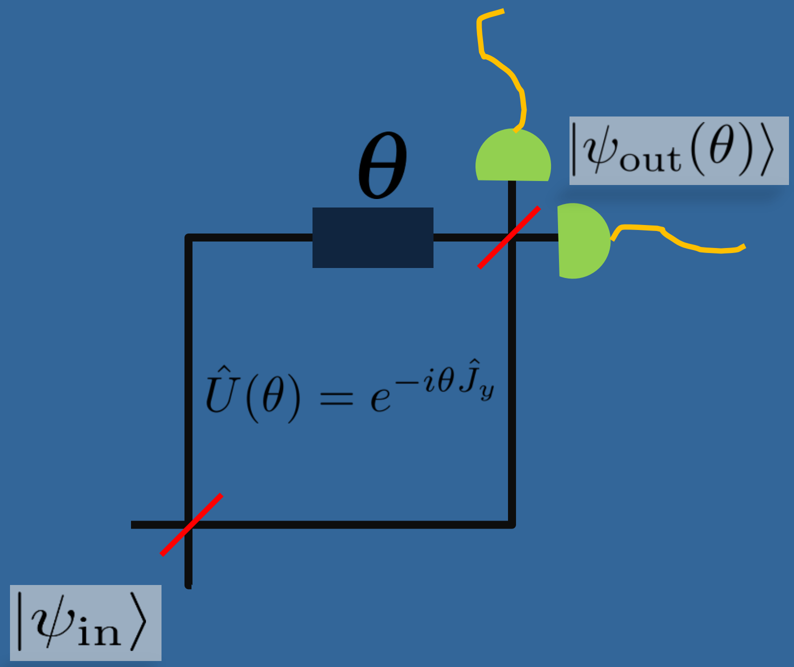
J. Ch. Arxiv:1108.2785

$$F_1 = \int dx \, \frac{1}{\rho(x|\theta)} \left(\frac{\partial \rho(x|\theta)}{\partial \theta} \right)^2$$

$$C = \int dx \int dy \, G^{(2)}(x, y|\theta) \cdot \partial_\theta \log \rho(x|\theta) \cdot \partial_\theta \log \rho(y|\theta)$$

Mach Zehnder Interferometer

Estimate θ from the population imbalance



Spin-squeezing

$$\xi_n^2 = N \frac{\Delta^2 \hat{J}_z}{\langle \hat{J}_x \rangle^2}$$

Kitagawa & Ueda PRA 1993

$$\hat{J}_x = \frac{1}{2}\left(\hat{a}^\dagger \hat{b} + \hat{a}\hat{b}^\dagger\right)$$
$$\hat{J}_y = \frac{1}{2i}\left(\hat{a}^\dagger \hat{b} - \hat{a}\hat{b}^\dagger\right)$$
$$\hat{J}_z = \frac{1}{2}\left(\hat{a}^\dagger \hat{a} - \hat{b}^\dagger \hat{b}\right)$$

$$\Delta^2 \theta_{\rm ML} = \frac{1}{m}\frac{\xi_n^2}{N}$$

$$\xi_n^2 = 1$$

Shot-noise

$$\xi_n^2 = \frac{1}{N}$$

Heisenberg

V. Giovanetti, S. Loyd and L. Maccone, Science **306**, 1330 (2004)
L. Pezze' and A. Smerzi, PRL **102**, 100401 (2009)

MZI continued

On the other hand…

Use the formula for the estimation from the density: $\Delta^2\theta_{\mathrm{ML}} = \frac{1}{m}\left(\frac{1}{F_1} + \frac{C}{F_1^2}\right)$

$$F_1 = \int dx \, \frac{1}{\rho(x|\theta)} \left(\frac{\partial \rho(x|\theta)}{\partial \theta}\right)^2 \qquad C = \int dx \int dy \, G^{(2)}(x,y|\theta) \cdot \partial_\theta \log \rho(x|\theta) \cdot \partial_\theta \log \rho(y|\theta)$$

Use the two-mode input state: $|\psi_{\mathrm{in}}\rangle = \sum_{n=0}^{N} c_n |n, N-n\rangle$

Take the MZI evolution operator $\hat{U}(\theta) = e^{-i\theta \hat{J}_y}$

Using $|\psi_{\mathrm{out}}(\theta)\rangle$ calculate ρ and $G^{(2)}$

and get… $\Delta^2\theta_{\mathrm{ML}} = \frac{1}{m}\frac{\xi_n^2}{N}$

Estimation from the interference pattern

BEC trapped in a double-well potential

θ

$$\hat{\Psi}(x) = \psi_a(x)\hat{a} + \psi_b(x)\hat{b}$$

- Imprint the phase

$$\hat{\Psi}(x|\theta) = \psi_a(x)\hat{a} + \psi_b(x)e^{i\theta}\hat{b}$$

- Open the trap

- Detect separate atoms or…
- do the least square fit to the density

Interference pattern continued

Use: $$\Delta^2\theta_{\mathrm{ML}} = \frac{1}{m}\left(\frac{1}{F_1} + \frac{C}{F_1^2}\right)$$

Obtain the sensitivity

$$\Delta^2\theta_{\mathrm{ML}} = \frac{1}{m}\frac{1}{N}\left(\xi_\phi^2 + \frac{\sqrt{1-\nu^2}}{\nu^2}\right)$$

Phase squeezing

$$\xi_\phi^2 = N\frac{\Delta^2\hat{J}_y}{\langle\hat{J}_x\rangle^2}$$

Grond NJP 2010

Fringe visibility

$$\nu = \frac{2}{N}\langle\hat{J}_x\rangle$$

Compare with MZI

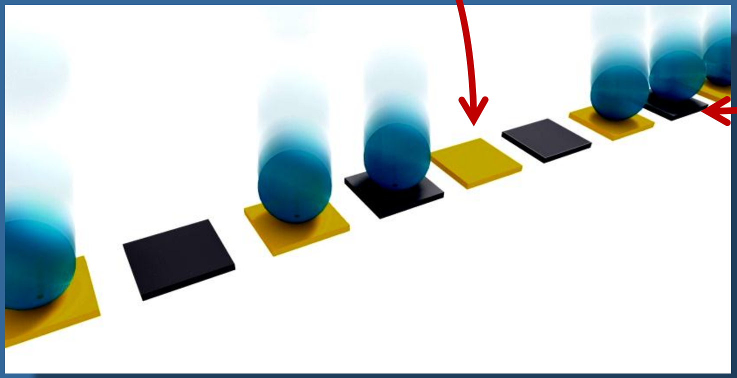
$$\Delta^2\theta_{\mathrm{ML}} = \frac{1}{m}\frac{\xi_n^2}{N}$$

$$\Delta^2\theta_{\mathrm{ML}} = \frac{1}{m}\frac{1}{N^2}$$

$$\Delta^2\theta_{\mathrm{ML}} = \frac{1}{m}\frac{2}{N^{\frac{4}{3}}}$$

Detection imperfections

Finite resolution $\Delta x = \dfrac{1}{10}$ th of the fringe



Finite efficiency $\eta = 90\%$

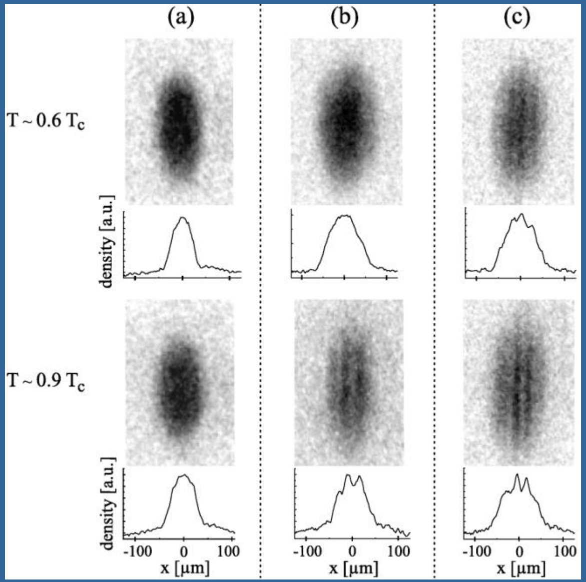
$$\Delta^2\theta_{\mathrm{ML}} = \frac{1}{m}\frac{2}{N^{\frac{4}{3}}}$$

$$\Delta^2\theta_{\mathrm{ML}} = \frac{1}{m}\frac{2}{N^{1.2}}$$

Still sub shot-noise

Estimation of the temperature

Far field image of a quasi-BEC



Dettmer PRL 2001

Approximate description:

$$\Psi_{\mathrm{qBEC}}(\vec{r}) = \phi_{\mathrm{GP}}(\vec{r})e^{i\varphi(z)}$$

Solution of the pure (*T*=0 K) GPE

Phase fluctuations resulting from thermal occupation of Bogoliubov modes

The goal: estimate T from a fit to the density

Petrov PRL 2001

Precision of temperature estimation

Use parameters of Palaiseau group:

$$N = 10^5 \ {}^4\mathrm{He}^*$$
$$m = 6.65 \times 10^{-27}\,\mathrm{kg}$$
$$a_s = 7.5 \times 10^{-9}\,\mathrm{m}$$
$$\omega_z = 2\pi \times 7.5\,\frac{1}{\mathrm{s}}$$
$$\omega_r = 200 \times \omega_z$$

We need:

$$\Delta^2 T = \frac{1}{F_1} + \frac{C}{F_1^2}$$

with

$$F_1 = \int d^3k \frac{1}{\rho(\vec{k}|T)} \left(\frac{\partial \rho(\vec{k}|T)}{\partial T} \right)^2$$

$$C = \int d^3k \int d^3k'\, G^{(2)}(\vec{k}, \vec{k}'|T) \partial_T \log\left[\rho(\vec{k}|T)\right] \partial_T \log\left[\rho(\vec{k}'|T)\right]$$

One realization

$$\Psi_{\mathrm{qBEC}}(\vec{r}) = \phi_{\mathrm{GP}}(\vec{r}) e^{i\varphi(z)} \xrightarrow{\text{F.T.}} \Psi_{\mathrm{qBEC}}(\vec{k}) \xrightarrow{\text{repeat}} \text{to get}$$

Precision of temperature estimation

Result:

$$\Delta T = \sqrt{\frac{1}{F_1} \cdots}$$

$$T_{\mathrm{est}} = T_{\mathrm{fit}} \pm \frac{1}{\sqrt{m}} \Delta T$$

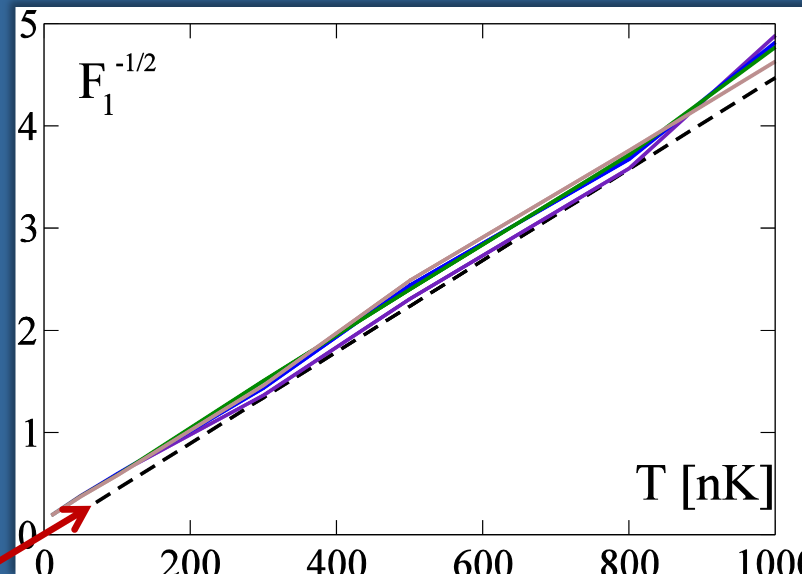
Theory:

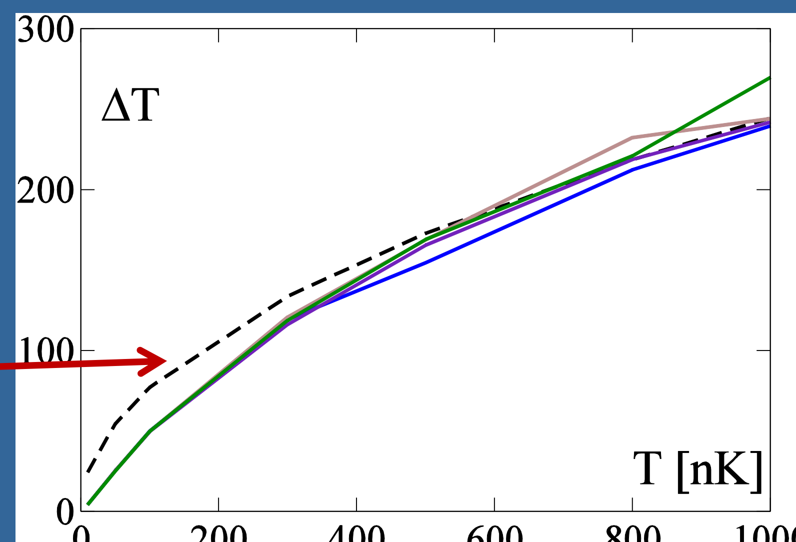
$$\Delta T \propto T$$

Theory:

$$\Delta T \propto \sqrt{T}$$

For $m = 100,\ \Delta T = 8\,\mathrm{nK}\ @\ 200\,\mathrm{nK}$





Summary

1. Precision of estimation from the density depends on $G(2)$

2. For the Mach-Zehnder Int. up to Heisenberg scaling

3. Interference pattern – also sub shot-noise

4. Possible estimation of T from a fit to the density of a qBEC

Thank you!